



ΠΑΜΙΒΙΑ UNIVERSITY
OF SCIENCE AND TECHNOLOGY
FACULTY OF HEALTH, APPLIED SCIENCES AND NATURAL RESOURCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BAMS	LEVEL: 6
COURSE CODE: PBT602S	COURSE NAME: Probability Theory 2
SESSION: JULY 2022	PAPER: THEORY
DURATION: 3 HOURS	MARKS: 100

SUPPLEMENTARY / SECOND OPPORTUNITY EXAMINATION QUESTION PAPER	
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MODERATOR:	Prof R. KUMAR

<p style="text-align: center;">INSTRUCTIONS</p> <ol style="list-style-type: none">1. There are 5 questions, answer ALL the questions by showing all the necessary steps.2. Write clearly and neatly.3. Number the answers clearly.4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculator

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1 [12 marks]

- 1.1. Briefly explain the following:
- 1.1.1. Boolean algebra $\mathfrak{B}(S)$ [2]
 - 1.1.2. Measure on a $\mathfrak{B}(S)$ algebra [2]
- 1.2. Show that if m is a measure on $\mathfrak{B}(S)$ and $c \geq 0$, then cm is a measure, where $(cm)(A) = c \cdot m(A)$ [4]
- 1.3. Let $S = \{1, 2, 3\}$, then find:
- 1.3.1. Power set, $\mathcal{P}(S)$ [2]
 - 1.3.2. size of $\mathcal{P}(S)$ [2]

Question 2 [17 marks]

- 2.1. An insurance company offers its policyholders a number of different premium payment options. For a randomly selected policyholder, let X be the number of months between successive payments. The cumulative distribution function of X is

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 0.4, & \text{if } 1 \leq x < 3, \\ 0.6, & \text{if } 3 \leq x < 5, \\ 0.8, & \text{if } 5 \leq x < 7, \\ 1, & \text{if } x \geq 7. \end{cases}$$

- 2.1.1. Use $F(x)$ to compute $P(3 \leq X \leq 5)$. [2]
 - 2.1.2. Find is the probability distribution function/probability mass function of X ? [3]
- 2.2. Let X and Y be a jointly distributed continuous random variable with joint p.d.f. of

$$f_{XY}(x, y) = \begin{cases} \frac{6}{5}(x + y^2), & \text{for } 0 \leq x \leq 1; 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- 2.2.1. Show that marginal p.d.f. of X , $f_X(x) = \frac{6}{5}\left(x + \frac{1}{3}\right)I_{(0, 1)}(x)$ [2]
- 2.2.2. Find $P(Y \geq 0.15|X = 0.25)$ [5]

- 2.3. Suppose that the joint p.d.f. of two continuous random variables X and Y is given by

$$f_{XY}(x, y) = \begin{cases} 12x, & 0 < y < x < 1; 0 < x^2 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the marginal p.d.f. of Y . [3]

- 2.4. The average weight of individuals in city A was 95kg with standard deviation of 10. If the city contains 100, 000 residents, what is the minimum number of individuals with a weight between 70kg and 120kg? [2]

QUESTION 3 [30 marks]

3.1. If X and Y are linearly related, in the sense that $Y = aX + b$, where $a > 0$, then show that

$$\rho_{XY} = 1. \quad [5]$$

3.2. Let X_1, X_2, \dots, X_n be independently and identically distributed with normal distribution with mean μ and variance σ^2 . Then show, using the moment generating function, that $Y = \sum_{i=1}^n x_i$ has a normal distribution and find the mean and variance of Y ?

Hint: $M_{X_i}(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$ [8]

3.3. Find the cumulant generating function for $X \sim N(\mu, \sigma^2)$ and hence find the first cumulant and the second cumulant. [7]

3.4. Let the random variable $X \sim N(\mu, \sigma^2)$. Find $E(X)$ and $Var(X)$ using the characteristic function of X . HINT: $\phi_X(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}$ [10]

QUESTION 4 [26 marks]

4.1. A certain radioactive mass emits alpha particles from time to time. The time between emissions, in seconds, is random, with probability density function

$$f_Y(y) = \begin{cases} 0.5e^{-0.5y}, & \text{for } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

4.1.1. Find the 25th percentile of the time between emissions [3]

4.1.2. Find the median time between emissions [3]

4.2. If X is a random variable having a binomial distribution with the parameters n and p (i.e., $X \sim Bin(n, p)$), then

3.4.1. Show that the moment generating function of X is given by $M_X(t) = (1 - p(1 - e^t))^n$. Hint: $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$ [4]

3.4.2. Find the first moment about the origin using the moment generating function of X . [3]

4.3. Let random variables $X_k \sim Poisson(\lambda_k)$ for $k = 1, \dots, n$ be independent Poisson random variables. If we define another random variable $Y = X_1 + X_2 + \dots + X_n$, then find $\phi_Y(t)$. Comment on the distribution of Y based on your result. [Hint $\phi_{X_k}(t) = e^{\lambda_k(e^t - 1)}$] [7]

4.4. Let Y be continuous random variable with a probability density function $f(y) > 0$. Also, let $U = h(Y)$. If h is increasing on the range of a given random variable, then show that [6]

$$f_U(u) = f_Y(h^{-1}(u)) \frac{d}{du} h^{-1}(u)$$

QUESTION 5 [15 marks]

- 5.1. Let X_1 and X_2 have joint p.d.f. $f(x_1, x_2) = 2e^{-(x_1+x_2)}$ for $0 < x_1 < x_2 < 1$. Let $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Find the joint p.d.f. of Y_1 and Y_2 , $g(y_1, y_2)$ [9]
- 5.2. Suppose that X and Y are independent, continuous random variables with densities $f_X(x)$ and $f_Y(y)$. Then $Z = X + Y$ is a continuous random variable with cumulative distribution function $F_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$ [6]

=== END OF PAPER===

TOTAL MARKS: 100